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ON THE NUMBER OF GENERATORS OF A MODULE

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1. Introduction

Many interesting questions in K-theory and in the study of rings and groups concerns the minimum number of elements required to generate a module. We shall study the situation especially from the point of view of stability. In particular Theorem 3 generalizes results (Theorems 1 and 2) of Bass and Gruenberg on the subject.

Throughout we shall be concerned with finitely generated modules over $\Lambda = Z\pi$, the integral group-ring of a group π of finite order g. *Modules will be assumed to be free as abelian groups unless otherwise stated.*

Definitions. Given modules M and N

1) $\mu(M)$ is the minimum number of generators that M requires.

2) M is stably equivalent to N, $M \doteq N$, if for some integer k, $M \oplus \Lambda^k \cong N \oplus \Lambda^k$.

3) M is weakly equivalent to N, $M \sim N$, if there are projective modules P and R of the same (finite) rank such that $M \oplus P \cong N \oplus R$.

Recall that if P is projective, then its Z-rank is an integral multiple of g, the order of π (e.g. cf. [7, p. 193]). Thus rank $P = (Z\text{-rank } P)/g$.

Bass has proved the following (immediate from [7, p. 193]).

Theorem 1. *If $P \doteq \Lambda^k$ and $k \geq 2$, then $P \cong \Lambda^k$.*

An equivalent statement is that if $P \doteq \Lambda^k$ and $k \geq 2$ then $\mu(P) = k$, because if $\mu(P) = k$, then there is an epimorphism $\Lambda^k \to P$ which, since it is an epimorphism of free abelian groups of rank $k \cdot g$, is an isomorphism.

Theorem 1 cannot be improved because Swan [5] gives an example of a projective module I such that $I \oplus \Lambda \cong \Lambda \oplus \Lambda$, but $\mu(I) = 2$ so $I \not\cong \Lambda$ although $I \doteq \Lambda$. In this case $\pi = \langle x, y \mid yxy^{-1}x, y^2x^8 \rangle$ is the generalized quaternion group of order 32.

Another result, due to Gruenberg, concerns a presentation of π as F/N where F

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is a finitely generated free group. The abelianization, N^a, of N is a Λ-module. Fixing F does not in general fix N nor, necessarily does it fix N^a. Gruenberg proved, however, [4]

Theorem 2. *$\mu(N^a)$ depends only on F.*

If $F/N \cong \pi \cong F/\bar{N}$, as mentioned above, N^a is not necessarily isomorphic to $\bar{N}^a$, however, $N^a \doteq \bar{N}^a$. This follows from the existence of an exact sequence

$$(*) \qquad 0 \to N^a \to \Lambda^t \to \Lambda \to Z \to 0 \quad [8]$$

where t is the rank of F, and from the Schanuel Lemma [7]. Furthermore this exact sequence shows that Z-rank $N^a = (t-1)g + 1$. Thus $N^a \cong Z$ if and only if $t = 1$ and $\mu(N^a) > 1$ otherwise. Thus both Theorems 1 and 2 follow from the more general result.

Theorem 3. *If $M \doteq N$ and neither is cyclic, then $\mu(M) = \mu(N)$.*

We shall prove Theorem 3 in Section 2.

Notation. $\mu_f(M) = \max_{p|g} \mu(Z_pM)$, $\mu_0(M)$ is the minimum number of $\mathbf{Q}\pi$-generators required for $\mathbf{Q}M$.

Remarks. 1) By kM we mean $k \otimes_Z M$ for a ring k.

2) $p \mid g$ means all primes p dividing g, or if $g = 1$ choose any one prime p. (So if $\pi = 1$, $\mu_f = \text{rank}$.)

3) $\mu(M) \geq \mu_f(M) \geq \mu_0(M)$: clearly generators for M map to generators for Z_pM and $\mathbf{Q}M$; and generators for Z_pM lift to elements mapping to generators for $\mathbf{Q}M$.

Definition [4]. A Swan module is a module M with $\mu(M) = \mu_f(M)$.

Swan modules are the "good" modules from the point of view of this study because the number of generators can be observed from the local situation. Swan modules M always have the property that $\mu(M \oplus \Lambda) = \mu(M) + 1$. Note that Maschke's Theorem [3, p. 423] implies that $Z_p\pi$ and $\mathbf{Q}\pi$ are semi-simple for $p \nmid g$. $\mu(Z_pM) = \mu_0(M)$ for $p \nmid g$.

An example of a non-Swan module is any projective P which is not free. (There are many known examples.) By [7, p. 193] $P \cong \Lambda^{r-1} \oplus J$, where J is a projective ideal of Λ. $J \subset \Lambda$ may be chosen of index prime to any given integer. Thus $Z_pP \cong (Z_p\pi)^r$ for any prime p. Thus $\mu_f(P) = r = \text{rank } P$. If P is not free then $\mu(P) > r$. We shall prove in Section 3 that this is the way that non-Swan modules arise by showing:

Theorem 4. *Every module M contains a Swan module $N \sim M$.*

Finally we shall use a result of Swan to observe the following result in Section 4:

Theorem 5. *If $\mathbf{Q}M \cong K \oplus \mathbf{Q}$ where $\mu_0(K) < \mu_0(M)$ then M is a Swan module.*

Throughout, $\mathbf{Q}$ or Z will always be modules with the trivial action of π.

Remark. Theorem 5 offers the general direction of Gruenberg's proof of Theorem 2; from $(*)$ earlier we see that $\mathbf{Q}N^a \cong \mathbf{Q}\pi^n \oplus \mathbf{Q}$ because $\mathbf{Q}\pi$ is semi-simple and thus $(*)$ splits totally. Clearly $\mu_0(\mathbf{Q}\pi^n \oplus \mathbf{Q}) = n+1 > \mu_0(\mathbf{Q}\pi^n)$ so N^a is a Swan module, thus $\mu(N^a)$ depends only on F.

I wish to thank Dick Swan for several useful discussions on this subject and related questions.

2. Proof of Theorem 3

The proof proceeds in three steps:

A) If $\mu(M \oplus \Lambda) \geqslant 3$ then $\mu(M \oplus \Lambda) = \mu(M) + 1$.

B) If $\mu(M) \geqslant 3$ then $\mu(M \oplus \Lambda^r) = \mu(M) + r$.

C) If $M \doteq N$ and either $\mu(M)$ or $\mu(N) \geqslant 3$ then $\mu(M) = \mu(N)$.

The result follows from C) because if M and N are not cyclic and neither $\mu(M)$ nor $\mu(N)$ is greater than 2, then $\mu(M) = 2 = \mu(N)$.

C) follows from B): if $M \oplus \Lambda^r \cong N \oplus \Lambda^r$ and $\mu(M) \geqslant 3$ then $\mu(N) \geqslant \mu(N \oplus \Lambda^r) - r = \mu(M \oplus \Lambda^r) - r = \mu(M) \geqslant 3$. Thus $\mu(N) = \mu(N \oplus \Lambda^r) - r = \mu(M)$.

B) follows from A) by induction. So we need only to prove part A): Let $k = \mu(M \oplus \Lambda)$ and let $f : \Lambda^k \to M \oplus \Lambda$ be an epimorphism. Let $p : M \oplus \Lambda \to \Lambda$ be the projection onto the second factor. Let $W = \ker pf$. Then the induced map $W \to M$ is onto. But $W \oplus \Lambda \cong \Lambda^k$ since pf splits. Thus $W \doteq \Lambda^{k-1}$. If $k \geqslant 3$, Theorem 1 implies $W \cong \Lambda^{k-1}$ so $\mu(M) = k-1$.

3. Proof of Theorem 4

First we need the following.

Proposition 1. *If N is a submodule of M of index n relatively prime to g, then $N \sim M$.*

Proof. Let $f : P \to M$ be any homomorphism such that P is projective and $f(P) + N = M$. Let $g : P \oplus N \to M$ be given by $g(p, n) = f(p) + n$. Let $R = \ker g$ with $\alpha : R \to P$, $\beta : R \to N$ the induced maps. Since $N \hookrightarrow M$ is a rational isomorphism, α is a rational isomorphism, hence α is a monomorphism. We thus consider $R \subset P$.

Claim: $P/R \cong M/N$.

Look at the short exact sequence of Λ-chain complexes $0 \to A_* \to B_* \to C_* \to 0$ where A_* is $0 \to N \hookrightarrow M$, B_* is $R \to P \oplus N \to M$, and C_* is $R \hookrightarrow P \to 0$. Since B_* is acyclic, $P/R = H_1(C_*) \cong H_0(A_*) = M/N$.

Since P/R is finite of order prime to g, for all primes $p \mid g$, $Z_pR \cong Z_pP$ is a free $Z_p\pi$-module. Thus R is projective [1, Theorem 8]. So R is weakly injective [2, p. 233]. R weakly injective means that if $R \subset M$ splits over Z, it splits over Λ. So the short exact sequence B_* splits as Λ-modules (since it splits as Z-modules). Thus $R \oplus M \cong P \oplus N$. Since M and N have the same Z-rank, P and R have the same Λ-rank, so $M \sim N$.

We can now prove Theorem 4:

Let $n = \mu_f(M)$ and let $a_1^p, a_2^p, \ldots, a_n^p$ be generators of Z_pM, for $p \mid g$. Let m_p be integers such that $m_p \equiv 1 \pmod p$ and $q \mid m_p$ for every other prime $q \mid g$. Let $a_i = \sum m_p a_i^p$. Then $a_1, \ldots, a_n$ are generators of Z_pM for all $p \mid g$. Let N be the submodule they generate. Then $N \subset M$ induces a Z_p-isomorphism for all $p \mid g$, thus M/N is of finite order prime to g. By the proposition, $N \sim M$. But $\mu(N) \leq n = \mu_f(M) = \mu_f(N)$ (because $Z_pM = Z_pN$) so N is a Swan module.

4. Related results

Recall that $M^\pi = \{m \in M \mid xm = m \text{ for all } x \in \pi\}$. We get

Proposition 2. *Let* $x \in M^\pi$ *and let* $N = M/(x)$. *Then* $\mu(M) \leq \max(2, \mu_f(M), \mu_0(N) + 1)$.

Proof. This is a reworking of Swan's Lemma 4.4 [6]. Let $n = \max(2, \mu_f(M), \mu_0(N) + 1)$. Since $n \geq 2$ and $n \geq \mu(Z_pM)$ for all $p \mid g$, n satisfies his conditions c) and a). Now $n \geq \mu_0(N) + 1$ so there is an epimorphism $\mathbf{Q}\pi^{n-1}$ onto $\mathbf{Q}N$. Since $\mathbf{Q}\pi$ is semi-simple all epimorphisms split. Thus $\mathbf{Q}\pi^{n-1} \cong \mathbf{Q}N \oplus K$, for some K. But now $\mathbf{Q}M \cong \mathbf{Q}N$ or $\mathbf{Q}N \oplus \mathbf{Q}$, depending on whether or not $x = 0$. Thus $\mathbf{Q}\pi^{n-1} \oplus \mathbf{Q} \cong \mathbf{Q}M \oplus K'$ where $K' \cong K \oplus \mathbf{Q}$ or K. Thus $\mathbf{Q}\pi^n \cong \mathbf{Q}M \oplus K' \oplus \mathbf{Q}\pi/\mathbf{Q}$. Since $\mathbf{Q}\pi/\mathbf{Q}$ contains a copy of every indecomposable $\mathbf{Q}\pi$-module except $\mathbf{Q}$, n satisfies condition b) of Swan's Lemma, whence $\mu(M) \leq n$.

We now prove Theorem 5 as follows:

If K is trivial, $\mathbf{Q}M \cong \mathbf{Q}$ so $M \cong Z$, a Swan module. If $K \neq 0$, then $\mu_f(M) \geq \mu_0(M) \geq 2$ and we can find $x \in M^\pi$, $x \neq 0$. Setting $N = M/(x)$, we see that $\mathbf{Q}N \cong K$. Thus $\mu_0(N) + 1 = \mu_0 K + 1 \leq \mu_0(M) \leq \mu_f(M)$. Thus $\max(2, \mu_f(M), \mu_0(N) + 1) = \mu_f(M)$. But $\mu(M) \geq \mu_f(M)$ always. So by Proposition 2, $\mu(M) = \mu_f(M)$.

In Section 1 we gave Swan's example of the failure of cancellation due to the fact that $I \doteq \Lambda$ but $\mu(I) \neq 1$. We now give an example where $M \doteq N$ but $M \not\cong N$ and M and N are Swan modules (so that $\mu(M) = \mu(N)$).

Example. Let I be Swan's example as in Section 1. $\mathrm{Hom}_\Lambda(Z, I) \cong Z$. Let $\mu : Z \to I$ be a generator. It is a monomorphism. By the techniques of [5], we see that $I/Z \ncong \Lambda/Z$ although $I/Z \stackrel{\wedge}{=} \Lambda/Z$ (where $Z \subset \Lambda$ is the trivial submodule generated by $\Sigma_{z\in\pi} z$). Let $^* = \mathrm{Hom}_\Lambda(-, \Lambda)$, an involution which preserves short exact sequences (proved in Proposition 3, below). Let $M = (I/Z)^*$ and $N = (\Lambda/Z)^*$. $M \stackrel{\wedge}{=} N$ but $M \ncong N$. M and N are respectively the kernels of epimorphism $I^* \to Z$ and $\Lambda \to Z$. Thus N is isomorphic to the augmentation ideal of Λ so $\mu(N) = 2$. $\mu(Z_2N) = 2$ so N is a Swan module. Thus $\mu(M) \geqslant \mu(M \oplus \Lambda) - 1 = \mu(N \oplus \Lambda) - 1 = 2$ so by Theorem 3 $\mu(M) = 2$. $Z_2M \cong Z_2N$ so M is also a Swan module.

Proposition 3. *On the category of finitely generated Λ-modules which are Z-free, $^* = \mathrm{Hom}_\Lambda(-, \Lambda)$ preserves short exact sequences and ** is naturally equivalent to the identity.*

Proof. Since projective Λ-modules are weakly injective * turns Z-split monomorphisms into epimorphisms, hence preserves exactness of short exact sequences of Z-free Λ-modules.

Let M be a module, $\alpha_M : M \to M^{**}$ the natural transformation $\alpha_M(m)(f) = f(m)$. Let $0 \to A \to \Lambda^k \to M \to 0$ be exact. Observe that α_{Λ^k} is an isomorphism. But then the exact diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha_A"] & \Lambda^k \arrow[r] \arrow[d, "\alpha_{\Lambda^k}"] & M \arrow[r] \arrow[d, "\alpha_M"] & 0 \\
0 \arrow[r] & A^{**} \arrow[r] & \Lambda^k \arrow[r] & M^{**} \arrow[r] & 0
\end{tikzcd}$$

yields the fact that α_A is a monomorphism and α_M an epimorphism. But there is an exact sequence $0 \to C \to \Lambda^t \to A \to 0$ so α_A is an epimorphism. Hence α_A is an isomorphism so α_M is also.

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